Atmospheric Stirring Measured by Precipitation.

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Since on the average the water-content of the atmosphere is not increasing, the water which descends as precipitation must have been stirred up into the atmosphere. So we can compute the eddy-diffusivity from a knowledge of the mean precipitation and the mean vertical gradient of mass of water per mass of atmosphere.

Eddy-diffusivity is a statistical measure of the effect of circulatory motions which we cannot, or do not wish to, consider in detail. Since the mean values in this paper are taken over the whole globe, it follows (unfortunately) that even the largest circulatory motions between the pole and the equator are not considered in detail, and that their effect is included with the effect of the rising currents in cyclone or anti-cyclones, with cumulus-eddies, and with the smaller eddies due to dynamical instability, in producing the coefficients of eddy-diffusivity at which we shall arrive.

The General Equations for Stirring.

In Taylor's theory of atmospheric stirring,* the density of the atmosphere and the stirring coefficient are treated as independent of height.

Under these restrictions he arrives at the equation

$$\frac{\partial \chi'}{\partial t} = \mathrm{K} \frac{\partial^2 \chi'}{\partial h^2}, \qquad (1)$$

where t is time, h is height, K is the eddy-diffusivity, and χ' may be either the potential temperature, or the vapour pressure, or the horizontal velocity in a fixed azimuth.

The present paper deals with a range of height involving considerable variations of density and very large variations in the stirring, so that it is necessary to find an equation more general than (1). At the same time, it will be convenient to arrange to have a simple expression not only for $\partial\chi'/\partial t$, but also for the vertical flux.

Here we may usefully bear in mind the analogy with the conduction of heat in a solid. The total water in a portion of air, or the total entropy in it, are not altered by gently mixing it; and the same is true of its horizontal momentum in a fixed azimuth, if the mixing takes place without the action

* 'Phil. Trans.,' A, vol. 215, p. 3 (1915).

of an external horizontal force in this azimuth. If, then, Z stands for any one of these three quantities or for any other quantity which

Has its total, for a definite portion of air, unchanged by the internal rearrangement of that portion, (2)

Or by delay, (3)

then the rate of increase of the total Z in the portion must be due to Z flowing in over the sides. If we now take the portion to be a horizontal layer, and define the "upward flux" as

The ratio of amount of Z rising across a large horizontal surface in unit time to the area of this surface, (4)

then rate of increase of Z in the layer dh must be

$$-\frac{\partial}{\partial h}(\text{upward flux})\, dh. \tag{5}$$

If now χ, without the dash, be defined as the amount of Z per unit mass of atmosphere, then

$$\text{The amount of Z in } dh \text{ is } \chi\rho dh, \tag{6}$$

so that

$$\frac{\partial(\chi\rho)}{\partial t} = -\frac{\partial}{\partial h}(\text{upward flux}). \tag{7}$$

Now we can define a coefficient c such that

$$\text{Upward flux} = -c\frac{\partial \chi}{\partial h}, \tag{8}$$

and it will then be for observation to show whether c is a constant, as would be nice, or whether c depends on various factors, and in particular on $\partial\chi/\partial h$. But, if c is going to be useful, it must not have to be infinite when $\partial\chi/\partial h = 0$. Therefore we had better confine ourselves to conditions in which

$$\text{The upward flux vanishes when } \partial\chi/\partial h = 0. \tag{9}$$

Now this implies that

χ must be unchanged by the simple transportation of air to a different level. (10)

Now χ is the total Z per unit mass of a definite portion of air. So, as the mass is unchanged by transport, (10) implies that

The total Z in a definite portion of air must be unchanged by its removal to a different level. (11)

Fortunately, (11) is satisfied by the same three quantities which previously satisfied (2), so that (10) is satisfied when χ is either mass of water per unit

of atmosphere, entropy per unit mass, or horizontal momentum per unit mass (better known as velocity), in a fixed azimuth. (11A)

If, in place of (8), we had made the flux proportional to $c' \partial(\rho\chi)/\partial h$ where $\rho\chi$ is the total amount of Z *per volume* instead of per mass of atmosphere then, to avoid an infinite value of c', the flux would have to vanish when $\partial(\rho\chi)/\partial h$ vanished. That would not be possible for the three given meanings of Z, because rising air changes its volume. So it appears that we had better keep to the definition (8).

Inserting (8) in (7), we have

$$\frac{\partial(\rho\chi)}{\partial t} = \frac{\partial}{\partial h}\left(c\frac{\partial\chi}{\partial h}\right). \tag{11B}$$

Now the left-hand side of (11B) is equal to $\rho\partial\chi/\partial t + \chi\partial\rho/\partial t$, and by the equation of continuity of mass,

$$\chi\partial\rho/\partial t = -\chi \operatorname{div} m, \tag{12}$$

where m is the vector momentum per unit volume.

Let us now suppose that any time change of mean density of the layer is either negligible, as will usually be the case, or else that it is separately accounted for. Then (11B) reduces to

$$\frac{\partial\chi}{\partial t} = \frac{\partial}{\rho\partial h}\left(c\frac{\partial\chi}{\partial h}\right) = -g\frac{\partial}{\partial p}\left(c\frac{\partial\chi}{\partial h}\right), \tag{13}$$

since

$$dp = -g\rho\, dh. \tag{14}$$

Equation (13) is in effect identical with one mentioned by G. I. Taylor for the case of potential temperature.*

On the right hand side of (13) there are two variables, p and h, both used as measures of height. It would be neater to use only one such measure. Therefore, as the dp is indispensable, let us replace the dh from (14). At the same time, in order to get a single constant, let us put $g^2\rho c = \xi$, then (13) becomes

$$\frac{\partial\chi}{\partial t} = \frac{\partial}{\partial p}\left(\xi\frac{\partial\chi}{\partial p}\right), \tag{15}$$

where χ may have any one of the meanings (11A). Here ξ is left to be determined by observation, but we know beforehand that it is finite when $\partial\chi/\partial p = 0$.

The flux is given by (5), (8), and (14) that is to say, if χ is the amount of Z per unit mass of atmosphere, then the amount of Z rising across a large horizontal surface per area per time is

$$+\frac{\xi}{g}\frac{\partial\chi}{\partial p} \quad \text{or, equivalently,} \quad -\frac{\xi}{g^2\rho}\frac{\partial\chi}{\partial h}. \tag{16}$$

* 'Roy. Soc. Proc.,' A, vol. 94, p. 140 (1918).

For the reasons already stated, it appears that it is the quantity ξ *of equations* (15) *and* (16) *which should be observed, tabulated, and remembered, in preference to* K *of equation* (1), *which should be regarded only as a stepping stone whereby to reach* ξ. The relation between K and ξ is found by equating the corresponding expressions for $\partial\chi/\partial t$, thus

$$K\frac{\partial^2\chi}{\partial h^2} = \frac{\partial}{\partial p}\left(\xi\frac{\partial\chi}{\partial p}\right). \tag{17}$$

If ξ should happen to be independent of height, equation (17) simplifies to

$$\xi = g^2\rho^2 K, \tag{18}$$

and in C.G.S. units the coefficient $g^2\rho^2$ is numerically equal to unity at an average height of 2 kilom. above mean sea level.*

The physical dimensions of ξ are

$$(\text{Mass})^2, (\text{length})^{-2}, (\text{time})^{-5}. \tag{19}$$

Note on Local Differences of Pressure in a Wind.

G. I. Taylor† arrived at equation (1) by considering small portions of air which wandered from levels at which they were average samples of the atmosphere and arrived at a new level, there to remain by mixing with the surroundings. When referring to the diffusion of momentum, he provides a proof to show that the local differences of pressure, which Stanton's observations have shown to be of considerable magnitude, do not nevertheless on the average affect the momentum of a sample during its journey. The local differences of pressure, therefore, leave equation (1) unaltered both in form and in the value of the eddy viscosity. But Taylor regrets that his proof of this applies only to motion in two dimensions.

In the present paper the difficulty does not arise in the same form. No attempt is here made to show theoretically that c is numerically the same for all three meanings of χ. That question is left to observation. Taylor, who originated this stimulating hypothesis, derived it from the consideration of "the mean vertical distance through which any portion of the atmosphere is raised or lowered while it forms part of an eddy till the time when it breaks off from it, and mixes with its surroundings." This distance would be difficult to define precisely, as mixing is a gradual process. Consequently, the hypothesis seems in need of empirical confirmation, and we come back again to observation.

What has been taken for granted in deriving equation (15), and has now

* Computer's Handbook, diagram to face Part II, p. 55.

† 'Phil. Trans.,' A, vol. 215 (1915).

to be examined more closely, is that (2), (3), and (9) are satisfied when Z is momentum and χ velocity, both in the same fixed horizontal direction. Now (3) is satisfied because the external forces on a horizontal lamina, due to local pressure differences, are normal to its surfaces and therefore cannot alter its horizontal momentum. Condition (2) is satisfied because forces between the parts cannot alter the momentum of the whole. In examining (9) we must distinguish between the mean velocity $\bar{v}$, which here plays the part of χ, and v', the deviation from the mean. The question is whether, if $\partial\bar{v}/\partial h$ vanish at any level, can there be an upward flux of momentum across that level? In other words can there be an eddy-shearing-stress where there is no rate-of-mean-shear? The converse, of course, occurs where there are no eddies. Or to put the question over again in another way: imagine two adjacent horizontal layers each say 100 metres thick and having the same mean velocity; would it be possible for the faster moving portions to sort themselves out and to flock together into one layer, leaving the slower moving portions in the other? If molecules did that sort of thing, the occurrence would be one of the exceedingly rare exceptions to the second law of thermodynamics. We may reasonably expect that there is a corresponding law, on an enlarged scale so to speak, applicable to the statistical mechanics of eddy motion.

The only other possibility seems to be that portions of air should wander in from quite distant regions, giving a part of the vertical flux of momentum proportional to $\partial^3\bar{v}/\partial h^3$ and a term in either (1) or (15) proportional to $\partial^4\bar{v}/\partial h^4$.

Precipitation in the Kilometre next above the Ground.

To come now from the general to the particular, let Z be mass of water, then χ is mass of water per mass of atmosphere and, following Hertz, will be denoted by μ. The mass of water in the form of vapour rising across one horizontal square centimetre per second is given by (16) on the average of a large horizontal area. And this must be equal to the average rainfall when the horizontal area is extended at that level to cover the whole globe. Accordingly we have

$$\frac{\xi}{g^2\rho}\frac{\partial\mu}{\partial h} = (\text{mean rate of precipitation}). \qquad (20)$$

The observational data for insertion in this equation, to find ξ, have been extracted from Hann's 'Lehrbuch der Meteorologie,' 3rd edition, to which the following paged references refer. According to Süring, μ in free air at a height of 1 kilom. has an average equal to 76 per cent. of its value at sea-level (p. 233). The mean value of μ at sea-level has been put equal to $10{\cdot}0 \times 10^{-3}$ which is its saturation value at 14·3° C. and 760 mm. Hg. (p. 221).

The temperature 14·3° C. is Dove's estimate of the mean temperature of the earth (p. 143). Consequently the average value of $\partial\mu/\partial h$ in the first kilometre is $-2{\cdot}4\times10^{-8}$ cm.$^{-1}$.

Next, the mean precipitation over the earth's surface is about 90 cm. year^{-1} according to Murray and Fritzche (p. 363). For insertion in the preceding equations, this must be expressed as $2{\cdot}9\times10^{-6}$ grm. cm.$^{-2}$ sec.$^{-1}$. This is the precipitation at the earth's surface. For comparison with the average value of $\partial\mu/\partial h$ in the first kilometre, we should use the precipitation at a height of half a kilometre. Unfortunately, there is not much instrumental evidence concerning change of precipitation with height, as elevated rain-gauges are so much disturbed by wind (pp. 319 to 323). However, as rain-clouds have an average height (p. 280) well above the height we are considering, and as rain, though colder than the air through which it falls, can hardly condense any moisture on to itself, owing to the large latent heat of steam (p. 320), and as, further, streaks of rain, descending from clouds, are sometimes seen to disappear before reaching the earth (p. 303), the evidence, as far as it goes, indicates that the precipitation does not vary much up to a height of 500 metres. The mean density at this height is about $1{\cdot}2\times10^{-3}$ grm. cm.$^{-3}$. Inserting the above mean values in the equation (20), we get $\xi = 1{\cdot}4\times10^{5}$ cm.$^{-2}$ grm.^{2} sec.$^{-5}$ as a mean value for the whole earth at a height of half a kilometre. This measure of diffusion is in remarkable agreement with the mean deduced by Taylor* from an analysis of temperatures on the Eiffel Tower; for Taylor found $K = 10{\cdot}0\times10^{4}$, whence on the assumption that ξ did not vary with height, equation (18) would give $\xi = g^2\rho^2\times10^{5} = 1{\cdot}45\times10^{5}$ C.G.S. units. It is also of the same order as the values of $g^2\rho^2$ (viscosity)/(density) found by Akerblom, Hesselberg, and Taylor by analyses of the change of wind with height. Thus Taylor's hypothesis, that the same constant applies to both cases, is so far confirmed.

To revert to the exchanges of moisture, as the precipitation does not vary much with height between 500 metres and the ground, it follows from (20) that the product of ξ and $\partial\mu/\rho\partial h$ must, on the average, be independent of height in this range. This statement will not apply to points within the vegetation film, because there transpiration will provide a source of $\partial\mu/\partial t$ which has not been considered in forming the equations. Now, just above the vegetation film, $-\partial\mu/\partial h$ is sometimes strikingly greater than the mean value $+2{\cdot}4\times10^{-8}$ cm.$^{-1}$, which has been taken to hold at a height of 500 metres. For instance, Hill's results (p. 238) at Allahabad in latitude $25\frac{1}{2}^{\circ}$ N. indicate an annual mean value of $\partial\mu/\partial h = -66\times10^{-8}$ cm.$^{-1}$ between 1·2 metre and 14·0 metres above the ground. Again, in the metre

* 'Roy. Soc. Proc.,' A, vol. 94 (1918).

immediately above a grass moor, I have observed $\partial\mu/\partial h = -1500 \times 10^{-8}$ cm.$^{-1}$ on an almost calm sunny March afternoon in latitude 49° N., when the surface soil was moist. This observation was taken with an aspirated wet-and-dry psychrometer. It is, of course, only an instance, perhaps an extreme one, but it suggests that the mean value of ξ in the first metre of height may be as low as 1000 or even less, which is interesting, because the values of ξ given by the observation of small floating objects to be published presently,* also indicate that ξ is much less near the earth than at a height of 1 kilometre.

Precipitation at a Height of 8·5 *Kilometres.*

Next, it seems desirable to attempt to estimate the stirring coefficient ξ from the precipitation at great heights; for, although the estimate will inevitably be a very uncertain one, it offers the only indication available at the moment of the order of this important quantity. The precipitation at 8·5 kilom. probably does not consist of anything except the slow descent of the cloud itself. The evidence for this statement is that the size of the precipitating particles is observed to decrease with height, so that, at the highest mountain stations, at a height still much below 8·5 kilom., there is observed nothing larger than drizzle and fine powdery snow (p. 358).† Also that such a cloud-name as "cirro-nimbus" is not to be found in a list of cloud-forms (pp. 269 to 273). We can calculate the velocity V_H with which the cloud descends, by Stokes' formula, when we know the size of the ice particles. This has been measured by Pernter, by observation of coronæ, and he has found the diameter to range from $0{\cdot}5 \times 10^{-3}$ cm. to $2{\cdot}0 \times 10^{-3}$ cm. (p. 257). It will be convenient to make the following calculations separately for these two extreme diameters. Now Stokes' formula for the steady descent of a very small sphere is

$$V_H = -\frac{g}{18}\frac{(\text{Diameter})^2\{(\text{density of sphere})-(\text{density of fluid})\}}{(\text{Molecular viscosity of the fluid})}.$$

The molecular viscosity of the air at the temperature 230° A, which prevails at 8·5 kilom., is about $1{\cdot}48 \times 10^{-4}$ C.G.S. and accordingly, for ice spheres of density 0·9, the terminal velocity is $V_H = -0{\cdot}33 \times 10^{-6} \times (\text{diameter})^2$, so that, for Pernter's limiting diameters, $V_H = -0{\cdot}08$ cm. sec.$^{-1}$ and $-1{\cdot}3$ cm. sec.$^{-1}$.

In comparison with these velocities due to gravity, those due to the electric charge and potential gradient are probably quite negligible, for at a height of 8·5 kilom. the mean potential gradient is probably only about

* In "Weather Prediction by Numerical Process," Cambridge University Press.

† Paged references here are to Hann's 'Meteorologie,' 3rd edition.

0·04 volts per centimetre (p. 740), while the mobility of the large atmospheric ions has been found by Langevin and Pollock to be less than 10^{-3} cm.^{2} sec.$^{-1}$ volt^{-1} (Kaye and Laby's tables), and the mobility of charged cloud particles would probably be even smaller.

On the other hand, the vertical velocity of the air may sometimes exceed the downward velocity of the particles through the air. The vertical velocity of the air, or more strictly its product with density, is here treated as part of the eddy motion, because its mean value over a level surface, including the whole globe, is zero. The air motion, therefore, does not affect our measure of precipitation, at least if horizontal variation of density can be neglected.

Suring's table for μ stops at 8 kilom. Taking μ as $10{\cdot}0 \times 10^{-3}$ at the earth's surface as before, it follows that mean $\partial\mu/\partial h = -0{\cdot}3 \times 10^{-8}$ cm.$^{-1}$ at 7·5 kilom. (p. 233). However, it is found later that we require $\partial\mu/\partial h$ at the height of the centre of mass of cirrus, cirro-stratus and cirro-cumulus clouds, a height which is about 8·5 kilom. Making a short extrapolation from Süring's table, it is found that $\partial\mu/\partial h = -0{\cdot}2 \times 10^{-8}$ cm.$^{-1}$ at 8·5 kilom. Even if this figure were ten times in error, the stirring deduced from it would still be of interest.

The density of the air at 8·5 kilom. has a mean value of about $0{\cdot}50 \times 10^{-3}$ grm. cm.$^{-3}$.*

The mass of ice per unit, horizontal area of cirrus, cirro-stratus, and cumulus clouds has been estimated in another paper† as from

$$0{\cdot}22 \times 10^{-3} \text{ to } 0{\cdot}9 \times 10^{-3} \text{ grm.-cm.}^{-2},$$

according as the particle diameter was

$$0{\cdot}5 \times 10^{-3} \text{ or } 2{\cdot}0 \times 10^{-3} \text{ cm.}$$

these being the limits of the range observed by Pernter (Hann, p. 257). It will be convenient to follow these limits throughout the argument. The corresponding velocities of the particles in still air are, from Stokes' formula:—

$$V_H = -0{\cdot}08 \text{ or } -1{\cdot}3 \text{ cm.-sec.}^{-1}.$$

Now the cloud, when it occurs, often consists of thin layers separated by clear air. So, to find a mean mass of ice per volume, we must take, as the thickness of the layer, not the thickness of the cloud itself, but the mean range within which it occurs, equal to, say, twice the "probable variation" of its height. Now the extreme observed heights for these three kinds of cloud are 20 kilom. and 2 kilom. (p. 281). A quantity which varies at

* Sir W. N. Shaw, 'Principia Atmospherica,' Table III.

† "Measurement of Water in Clouds," *infra*, p. 19.

random according to the law of error, as we may suppose the height of the cloud to do within this range, commonly has a "probable variation" equal to about 1/8 of the total range, which in this case is 18 kilom. So we must imagine the ice as spread out in a total range of height of about 4·5 kilom. Thus, in the parts of the sky where these clouds are present, the mass of ice per volume would be respectively $0{\cdot}05 \times 10^{-8}$ and $0{\cdot}20 \times 10^{-8}$ grm. cm.$^{-3}$ for Pernter's limiting diameters of particles. Next we require the mean fraction of the sky of the whole globe, which is covered by cirrus, cirro-stratus, or cirro-cumulus. This fraction will be less than the mean cloudiness observed at mountain stations at heights of 2 kilom. or 3 kilom., for the latter includes also the alto-cumulus and alto-stratus, as well as some cumulus and special mountain clouds.

Some mean annual cloudinesses are set out below:—

Station.	Height.	Latitude.	Fractional cloudiness.	Reference to Hann.
	km.			
Theodul pass	3·3	46° N.	0·54	Page 293
Pic du Midi	2·9	42° N.	0·48	,, 293
Blue Hill	—	42° N.	0·45	,, 288
Mexican tableland ...	2·2	20·5° N.	0·48	,, 294
Peruvian tableland ...	2·4	16° S.	0·37	,, 292

On the average we shall probably not be far wrong if we take 0·3 of the globe's sky to be covered by cirrus, cirro-stratus, or cirro-cumulus. It follows then that the mean mass of ice per volume of atmosphere at a height of 8·5 kilom. is respectively

$$= 0{\cdot}015 \times 10^{-8} \text{ and } 0{\cdot}060 \times 10^{-8} \text{ grm. cm.}^{-3}$$

for Pernter's limiting diameters of particles. Inserting this, together with $\rho = 0{\cdot}50 \times 10^{-3}$ grm. cm.$^{-3}$, $\partial\mu/\partial h = 0{\cdot}2 \times 10^{-8}$ cm.$^{-1}$, and V_H, respectively $-0{\cdot}08$ cm. sec.$^{-1}$ and $-1{\cdot}3$ cm. sec.$^{-1}$, in the formula which follows from equation (20), namely in

$$\xi = g^2\rho \frac{qV_H}{\partial\mu/\partial h} \tag{21}$$

there results $\xi = 3$ or 180 cm.$^{-2}$ grm.^{2} sec.$^{-5}$ according as we take the smallest or the largest ice particles observed by Pernter; for ξ deduced in this way is proportional to the cube of the diameter of the ice particle. There have been many uncertainties in this calculation of ξ at 8·5 kilom., but even its upper limit, $\xi = 180$, is so very much smaller than the value $\xi = 140000$ at a height of 0·5 kilom. as to make it certain that ξ diminishes greatly with

height in the upper part of the troposphere. In the stratosphere ξ must of course be so small as not to disturb the equality of temperature.

Summary.

The equation for diffusion is investigated in the general case in which the atmospheric density and the degree of turbulence must both be regarded as varying with height, and it is found that the stirring is best measured by the coefficient ξ in the equation

$$\frac{\partial \chi}{\partial t} = \frac{\partial}{\partial p}\left(\xi \frac{\partial \chi}{\partial p}\right)$$

in preference to K in $\partial\chi/\partial t = \mathrm{K}\partial^2\chi/\partial h^2$.

Here p is pressure, h the height, and χ may be either entropy* per unit mass, or mass of water per unit mass, or horizontal momentum per unit mass (better known as velocity) in a fixed azimuth.

From estimates of precipitation and of vertical gradient of water per mass of atmosphere, as means over the whole globe, it is found that ξ has the following mean values :—

Height.	
8500 metres	3 to 180 cm.$^{-2}$ grm.^{2} sec.$^{-5}$
500 „	140000
0·5 „	Possibly as low as 1000, or even less.

The value at 500 metres is in fair agreement with ξ as deduced from K calculated by Akerblom, Hesselberg, and Taylor for heights of a few hundred metres. The values of ξ at other levels are remarkably smaller.

* Mr. W. H. Dines points out in regard to entropy, that the statement (2) is only an approximation, for mixing depends ultimately on conduction or radiation, and these are irreversible processes which cause the total entropy to increase. However, when the initial difference of temperature between the ingredients is a very small fraction of the absolute temperature of the mixture, as in the case considered, the error committed must be very small. Potential temperature does not involve this error.